

# Generation of quantum entanglement between three level atoms via $n$ coupled cavities

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### Abstract

Based on two-photon exchange interaction between  $n$  coupled optical cavities each of them containing a single three level atom, the  $n$ -qubit and  $n$ -photonic state transfer is investigated. In fact, following the approach of Ref.[1], we consider  $n$  coupled cavities instead of two cavities and generalize the discussions about quantum state transfer, photon transition between cavities and entanglement generations between  $n$  atoms. More clearly, by employing the consistency of number of photons (the symmetry of Hamiltonian), the hamiltonian of the system is reduced from  $3^n$  dimensional space into  $2n$  dimensional one. Moreover, by introducing suitable basis for the atom-cavity state space based on Fourier transform, the reduced Hamiltonian is block-diagonalized, with 2 dimensional blocks. Then, the initial state of the system is evolved under the corresponding Hamiltonian and the suitable times  $T$  at which the initially unentangled atoms, become maximally entangled, are determined in terms of the hopping strength  $\xi$  between cavities.

**Keywords:** coupled cavities, two-photon exchange, hopping strength, three level atoms, generation of entanglement, excitation and photon transfer, Fourier transform

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# 1 Introduction

The quantum communication between several parts of a physical unit, is a crucial ingredient for many quantum information processing protocols [2]. Schemes for the transfer of quantum information and the generation and distribution of entanglement have been designed and implemented, in the past years, in a number of physical systems (see for example [3]-[12]). Atoms and ions are particularly considered as tools for storing quantum information in their internal states. Naturally, photons represent the best qubit carrier for fast and reliable communication over long distances [13, 14]. Recently, using photons in order to achieve efficient quantum transmission between spatially distant atoms has been considered in several works [1, 15, 16, 17, 18, 19]. The basic idea, is to utilize strong coupling between optical cavities and the atoms. On the other hand, due to the ability of quantum entanglement as a resource for several quantum information processing tasks such as quantum communication, and certain quantum cryptographic protocols, the creation of quantum entanglement naturally arises as goals in nowadays quantum control experiments in studying the nonclassical phenomena in quantum physics.

One of the known models in quantum optics describing the atom-field interaction is the Jaynes-Cummings Hamiltonian [20, 21]. In the study of three-level atoms, M. Alexanian and S. Bose [17] introduced a unitary transformation, whereby the three-level atom was reduced to a corresponding two-level atom of the Jaynes-Cummings type with two-photon instead of single-photon transitions. In Refs. [1, 18], entanglement properties of two and three atom-cavity systems in which the cavities are coupled via two-photon exchange interaction, was analyzed in detail. Such results could set the pathway towards massively correlated multiphoton nonlinear quantum optical systems [22, 23], which are rapidly developing modern subjects nowadays. The motivation of interest to such systems is their promise in quantum switching, quantum communication and computation and quantum phase transition applications.

In this paper, following the approach of Refs. [1, 17, 18], and introducing some suitable basis

in which the Hamiltonian of the system can be block-diagonalized, we generalize the discussions of Ref.[1] and Ref.[18] for two and three atom-cavity systems, to a system consisting of  $n$  coupled atom-cavity subsystems. More clearly, we consider a system of  $n$  spatially separated optical cavities, each containing a single three level atom, which are coupled to each other with two-photon exchange interaction. Our objective is to examine state transfer (atomic state exchange or photon transition) within photon and atom subsystems and to consider possible generation of the particle entanglement between the subsystems.

The organization of the paper is as follows. In section 2, the model describing a system of  $n$  identical atom-cavity subsystems is introduced. The main results of the paper such as block-diagonalization of the Hamiltonian of the system, solving the Shrödinger equation for time dependent probability amplitudes of the state of the system, and discussions about state transfer (atomic excitation or photon transitions) and entanglement generation between atoms or photons, are given in this section. Sections 3 and 4 are respectively concerned with the special cases of two and three identical coupled cavities. Paper is ended with a brief conclusion.

## 2 The Model: $n$ coupled cavities via two-photon exchange interaction

We will consider  $n$  identical cavities each containing one three-level atom, where the cavities are coupled via two photon hopping between them. In fact, we consider that the cavities are located at the nodes of the complete graph  $K_n$  with  $n$  nodes and each cavity interacts with all of the other cavities via two-photon exchange.

Let us first introduce the two-photon Hamiltonian obtained via an exact unitary transformation introduced in Ref. [17]:

$$H^{(i)} = \hbar\omega N^{(i)} + E_0^{(i)} + \hbar\mu\sigma_{ee}^{(i)} + \hbar\eta\sigma_{gg}^{(i)} + \hbar\lambda(\sigma_{eg}^{(i)}a_i^2 + \sigma_{ge}^{(i)}a_i^{2\dagger})$$

where, the operator

$$\hat{N}^{(i)} = a_i^\dagger a_i + \sigma_{ee}^{(i)} - \sigma_{gg}^{(i)} + 1$$

is a constant of motion for the  $i$ -th atom-cavity subsystem, i.e, we have  $[H^{(i)}, \hat{N}^{(i)}] = 0$  for each  $i = 1, 2, \dots, n$ . The operators  $a_i$  and  $a_i^\dagger$  are photon operators of the  $i$ -th cavity, and  $\sigma_{ab}^{(i)} = |a\rangle^{(i)(i)}\langle b|$ , for  $i = 1, 2, \dots, n$  denote the atomic transition operators for the  $i$ -th cavity referring to either the ground (g) or excited (e) state. Now, the Hamiltonian for the  $n$  cavities is given by:

$$H = \sum_{i=1}^n (H^{(i)} - H_0^{(i)}) + \hbar\xi \sum_{i,j=1; i < j}^n (a_i^{2\dagger} a_j^2 + a_j^{2\dagger} a_i^2), \quad (2-1)$$

where,

$$H_0^{(i)} = \hbar\omega(\hat{N}^{(i)} - 1) + (E_g + E_e)/2,$$

with  $E_g$ ,  $E_e$  being the energies of the ground and excited states, respectively. The last term in the Hamiltonian (2-1) is the two-photon exchange interaction between the cavities, characterized by the hopping rate  $\xi$ . The parameters  $E_0, \mu, \eta$  and  $\lambda$  are the free energies of the subsystems written in the notation of Ref. [17] and we do not need their clear definitions in the present paper. All of these parameters depend on the photon number in the corresponding cavities and so, on the cavity-mode intensity through the eigenvalues of the operator  $\hat{N}^{(i)}$ .

The operator  $\hat{N} = \sum_{i=1}^n \hat{N}^{(i)}$  commutes with the Hamiltonian (2-1) and so we can reduce the Hamiltonian to the subspace spanned with the eigenstates of  $\hat{N}$  and consider the time evolution of the states in this subspace. For a given eigenspace of  $\hat{N}$  with eigenvalue  $N$ , the maximum possible number of photons in a cavity is  $N$  when the corresponding atom is in the ground state, which occurs when there are no photons present in the other cavities and the atoms are also in the ground state. Then, the total number of photons in the system will be  $N$ . The constant number of total photons determines the subspace or the manifold in which the states evolve in time (the initial state of the system determines the constant number  $N$ ). We will consider the manifold with  $N = 2$ . In this case, each single atom-cavity system can take

one of the three possible states  $|g, 0\rangle$ ,  $|g, 2\rangle$  or  $|e, 0\rangle$ , and so, the total possible states that the system of  $n$ -cavities can take, are  $3^n$  states. Due to the consistency of total  $N = 2$ , the only possible states which we can have, are  $2n$  states instead of  $3^n$  ones. In fact, these  $2n$  states are eigenstates of  $\hat{N}$  with eigenvalue 2, and the  $3^n$ -dimensional Hamiltonian  $H$  is reduced to  $2n$ -dimensional one in the bases which span the eigenspace of  $\hat{N}$  with the corresponding eigenvalue 2. The bases states that span this subspace or manifold, are given by:

$$\begin{aligned} |c_i\rangle &= |g, 0\rangle \dots |g, 0\rangle \underbrace{|g, 2\rangle}_{i\text{-th}} |g, 0\rangle \dots |g, 0\rangle, \\ |a_i\rangle &= |g, 0\rangle \dots |g, 0\rangle \underbrace{|e, 0\rangle}_{i\text{-th}} |g, 0\rangle \dots |g, 0\rangle, \end{aligned}$$

for  $i = 0, 1, \dots, n-1$ . Indeed, these bases span the eigenspace of  $\hat{N}$  with eigenvalue 2, i.e., we have  $\hat{N}(\alpha|c_i\rangle + \beta|a_i\rangle) = 2(\alpha|c_i\rangle + \beta|a_i\rangle)$ . Therefore, the general time dependent state of the  $n$ -cavity system is given by

$$|\psi(t)\rangle = \sum_{i=0}^{n-1} (C_i(t)|c_i\rangle + A_i(t)|a_i\rangle). \quad (2-2)$$

Then, one can easily show that, by considering the order of bases as  $|c_0\rangle, |a_0\rangle, \dots, |c_{n-1}\rangle, |a_{n-1}\rangle$ , the Hamiltonian  $H$  takes the following direct product form

$$H = I_n \otimes \begin{pmatrix} 1 & \tan \theta_0 \\ \tan \theta_0 & \tan^2 \theta_0 \end{pmatrix} + 2\xi(J_n - I_n) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2-3)$$

where,  $I_n$  is  $n \times n$  identity matrix and  $J_n$  is the all one matrix of order  $n$ . The quantity  $\tan \theta_0$  is given by  $\tan \theta_0 = \frac{1}{\sqrt{2}r}$  with  $r = \frac{g_1}{g_2}$ , where  $g_1$  and  $g_2$  are the atom-photon coupling constants in the three-level atom. In writing the above equation, the dimensionless time  $[(E_0^+ - E_0^-) \cos^2 \theta_0]t/\hbar \rightarrow t$  and dimensionless hopping constant  $\hbar\xi/[(E_0^+ - E_0^-) \cos^2 \theta_0] \rightarrow \xi$  have introduced (see Ref.[1, 18] for the cases  $n = 2$  and  $n = 3$  coupled cavities), where  $E_0^+$  and  $E_0^-$  are eigenvalues associated with eigenvectors  $|\psi_0^+\rangle^{(i)} = \sin \theta_0|e, 0\rangle + \cos \theta_0|g, 2\rangle$  and  $|\psi_0^-\rangle^{(i)} = \cos \theta_0|e, 0\rangle + \sin \theta_0|g, 2\rangle$  of  $H^{(i)}$ , respectively.

It is well known that the matrix  $J_n$  has eigenvalues 0, and  $n$  (due to the fact that  $J_n^2 = nJ_n$ ), and is diagonalized by discrete Fourier transform  $F$  defined as  $F_{kl} := \frac{1}{\sqrt{n}}\omega^{kl}$  for  $k, l = 0, 1, \dots, n-1$ , where  $\omega = \exp(\frac{2\pi i}{n})$  is the  $n$ -th root of unity. Therefore, by introducing the new Fourier transformed bases  $\{|c_i\rangle', |a_i\rangle'\}_{i=0}^{n-1}$  as:

$$\begin{aligned} |c_l\rangle' &:= \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \omega^{li} |c_i\rangle, \\ |a_l\rangle' &:= \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \omega^{li} |a_i\rangle \end{aligned} \quad (2-4)$$

and considering the ordering  $\{|c_0\rangle', |a_0\rangle'; \dots; |c_{n-1}\rangle', |a_{n-1}\rangle'\}$ , the Hamiltonian (2-3) takes the following block diagonalized form:

$$H = I_n \otimes \begin{pmatrix} 1 & \tan \theta_0 \\ \tan \theta_0 & \tan^2 \theta_0 \end{pmatrix} + 2\xi \text{diag}(n-1, -1, \dots, -1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2-5)$$

where,  $\text{diag}(n-1, -1, \dots, -1)$  is the  $n \times n$  diagonal matrix with diagonal entries as  $n-1$  and  $-1$  respectively. Now, by using the Schrödinger equation of motion  $i\hbar \frac{\partial}{\partial t} |\psi\rangle = H|\psi\rangle$ , the equations of motion are given by:

$$i\dot{C}'_0 = [1 + 2\xi(n-1)]C'_0 + \tan \theta_0 A'_0;$$

$$i\dot{A}'_0 = \tan \theta_0 C'_0 + \tan^2 \theta_0 A'_0,$$

and

$$i\dot{C}'_l = (1 - 2\xi)C'_l + \tan \theta_0 A'_l;$$

$$i\dot{A}'_l = \tan \theta_0 C'_l + \tan^2 \theta_0 A'_l, \quad (2-6)$$

for  $l = 1, 2, \dots, n-1$ . The equations (2-6) can be exactly solved for any value of  $\tan \theta_0$ . Here we take the ratio of atomic couplings in the three level atoms as  $r = 1/\sqrt{2}$  so that we have  $\tan \theta_0 = 1$ . Substituting  $\tan \theta_0 = 1$  in (2-6) and solving the corresponding differential equations, one can obtain

$$C'_0(t) = \frac{e^{-i[1+\xi(n-1)]t}}{\sqrt{1+\xi^2(n-1)^2}} \{ [\sqrt{1+\xi^2(n-1)^2} \cos t \sqrt{1+\xi^2(n-1)^2} - i\xi(n-1) \sin t \sqrt{1+\xi^2(n-1)^2}] C'_0(0) -$$

$$\begin{aligned}
& i \sin t \sqrt{1 + \xi^2(n-1)^2} A'_0(0) \}, \\
A'_0(t) = & \frac{e^{-i[1+\xi(n-1)]t}}{\sqrt{1 + \xi^2(n-1)^2}} \{ [\sqrt{1 + \xi^2(n-1)^2} \cos t \sqrt{1 + \xi^2(n-1)^2} + i\xi(n-1) \sin t \sqrt{1 + \xi^2(n-1)^2}] A'_0(0) - \\
& i \sin t \sqrt{1 + \xi^2(n-1)^2} C'_0(0) \} \quad (2-7)
\end{aligned}$$

where, for  $l = 1, 2, \dots, n-1$  we obtain

$$\begin{aligned}
C'_l(t) &= \frac{e^{-i(1-\xi)t}}{\sqrt{1 + \xi^2}} \{ [\sqrt{1 + \xi^2} \cos t \sqrt{1 + \xi^2} + i\xi \sin t \sqrt{1 + \xi^2}] C'_l(0) - i \sin t \sqrt{1 + \xi^2} A'_l(0) \}, \\
A'_l(t) &= \frac{e^{-i(1-\xi)t}}{\sqrt{1 + \xi^2}} \{ [\sqrt{1 + \xi^2} \cos t \sqrt{1 + \xi^2} - i\xi \sin t \sqrt{1 + \xi^2}] A'_l(0) - i \sin t \sqrt{1 + \xi^2} C'_l(0) \}. \quad (2-8)
\end{aligned}$$

By using (2-4), one can obtain the time dependence of the coefficients  $C_i(t)$  and  $A_i(t)$  of the state of the system in (2-2) via the inverse Fourier transform as,

$$\begin{aligned}
C_i(t) &= \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \omega^{-li} C'_l(t), \\
A_i(t) &= \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \omega^{-li} A'_l(t). \quad (2-9)
\end{aligned}$$

It should be pointed out that, one can evaluate the probabilities associated with the state of the system as a superposition of atomic states  $|a_i\rangle$ , and that of photonic states  $|c_i\rangle$ , denoted by  $P_a(t)$  and  $P_c(t)$ , respectively. For instance, considering the initial state  $|\psi(0)\rangle = \frac{1}{\sqrt{n}}(|g, 2\rangle|g, 0\rangle \dots |g, 0\rangle + |g, 0\rangle|g, 2\rangle|g, 0\rangle \dots |g, 0\rangle + \dots + |g, 0\rangle \dots |g, 0\rangle|g, 2\rangle)$ , with initial conditions  $A_l(0) = 0$  and  $C_l(0) = \frac{1}{\sqrt{n}}$  for all  $l = 0, 1, \dots, n-1$ , with the aid of Eqs. (2-2) and (2-8), we obtain

$$\begin{aligned}
P_c(t) &= \sum_{l=0}^{n-1} |C_l(t)|^2 = \sum_{l=0}^{n-1} |C'_l(t)|^2 = \frac{1}{n[1 + \xi^2(n-1)^2]} \{ \xi^2(n-1)^2 + \cos^2 t \sqrt{1 + \xi^2(n-1)^2} \} + \\
& \quad \frac{n-1}{n(1 + \xi^2)} \{ \xi^2 + \cos^2 t \sqrt{1 + \xi^2} \}, \\
P_a(t) &= \sum_{l=0}^{n-1} |A_l(t)|^2 = \sum_{l=0}^{n-1} |A'_l(t)|^2 = 1 - P_c(t). \quad (2-10)
\end{aligned}$$

where, in the second equality in  $P_c(t)$  and that of  $P_a(t)$ , we have used the fact that the Fourier transform is unitary and so does not change the norm of vectors. The above result indicates



that, in the limit of large  $\xi \rightarrow \infty$ , we have  $P_c(t) \simeq 1$  for every time  $t$ , i.e., for large enough  $\xi$ , all of the atoms will be at their ground state  $|g\rangle$  at every time  $t$ .

## 2.1 Large and small hopping strengths

One should notice that for large values of the hopping strength, i.e.,  $\xi \gg$ , the evaluated coefficients  $C'_i(t)$  and  $A'_i(t)$  in (2-7) and (2-8) take the form

$$\begin{aligned} C'_0(t) &\simeq e^{-i[1+\xi(n-1)]t} \left\{ e^{-i\xi(n-1)t} C'_0(0) - \frac{i \sin \xi(n-1)t}{\xi(n-1)} A'_0(0) \right\}, \\ A'_0(t) &\simeq e^{-i[1+\xi(n-1)]t} \left\{ e^{i\xi(n-1)t} A'_0(0) - \frac{i \sin \xi(n-1)t}{\xi(n-1)} C'_0(0) \right\}, \\ C'_l(t) &\simeq e^{-i(1-\xi)t} \left\{ e^{i\xi t} C'_l(0) - \frac{i \sin \xi t}{\xi} A'_l(0) \right\}, \\ A'_l(t) &\simeq e^{-i(1-\xi)t} \left\{ e^{-i\xi t} A'_l(0) - \frac{i \sin \xi t}{\xi} C'_l(0) \right\}; \quad l = 1, 2, \dots, n-1. \end{aligned} \quad (2-11)$$

Neglecting also the second terms in the above approximations, we get

$$\begin{aligned} C'_0(t) &\approx e^{-2i\xi(n-1)t} C'_0(0), \\ C'_l(t) &\approx e^{2i\xi t} C'_l(0), \quad l = 1, \dots, n-1, \\ A'_l(t) &\approx A'_l(0), \quad l = 0, 1, \dots, n-1, \end{aligned}$$

and so by using (2-9), we obtain

$$\begin{aligned} C_l(t) &\approx \frac{e^{2i\xi t}}{n} \left\{ e^{-2i\xi n t} \sum_{k=0}^{n-1} C_k(0) + \sum_{k=0}^{n-1} \left[ \sum_{i=1}^{n-1} \omega^{(k-l)i} \right] C_k(0) \right\} = \frac{e^{2i\xi t}}{n} \sum_{k=0}^{n-1} (e^{-2i\xi n t} - 1 + n\delta_{k,l}) C_k(0), \\ A_l(t) &\approx A_l(0); \quad \text{for} \quad l = 0, 1, \dots, n-1 \end{aligned} \quad (2-12)$$

where, in the first relation we have used the fact that for the  $n$ -th root of unity  $\omega$ , we have  $\sum_{i=0}^{n-1} \omega^{(k-l)i} = n\delta_{k,l}$  and so  $\sum_{i=1}^{n-1} \omega^{(k-l)i} = n\delta_{k,l} - 1$ . The above results, are in correspondence with those of Refs. [1, 18] for the special cases  $n = 2$  and  $n = 3$ . Moreover, the relations (2-12) indicate that in the limit of large hopping strength, the state associated with the initially

unentangled atoms, i.e., the initial state with  $A_l(0) = 0$ , for  $l = 0, 1, \dots, n-1$ , remains effectively unentangled forever.

In the limit of small hopping  $\xi \ll 1$ , the equations (2-7) and (2-8) lead to the following coefficients  $C'_i(t)$  and  $A'_i(t)$

$$\begin{aligned}
C'_0(t) &\simeq e^{-i[1+\xi(n-1)]t} \{ [\cos t - i\xi(n-1) \sin t] C'_0(0) - i \sin t A'_0(0) \}, \\
A'_0(t) &\simeq e^{-i[1+\xi(n-1)]t} \{ [\cos t + i\xi(n-1) \sin t] A'_0(0) - i \sin t C'_0(0) \}, \\
C'_l(t) &\simeq e^{-i(1-\xi)t} \{ [\cos t + i\xi(n-1) \sin t] C'_l(0) - i \sin t A'_l(0) \}, \\
A'_l(t) &\simeq e^{-i(1-\xi)t} \{ [\cos t - i\xi(n-1) \sin t] A'_l(0) - i \sin t C'_l(0) \}; \quad l = 1, 2, \dots, n-1.
\end{aligned} \tag{2-13}$$

Now, by neglecting the terms proportional to  $\xi$ , the above approximations read as

$$\begin{aligned}
C'_0(t) &\approx e^{-i[1+\xi(n-1)]t} \{ \cos t C'_0(0) - i \sin t A'_0(0) \}, \\
A'_0(t) &\approx e^{-i[1+\xi(n-1)]t} \{ \cos t A'_0(0) - i \sin t C'_0(0) \}, \\
C'_l(t) &\approx e^{-i(1-\xi)t} \{ \cos t C'_l(0) - i \sin t A'_l(0) \}, \\
A'_l(t) &\approx e^{-i(1-\xi)t} \{ \cos t A'_l(0) - i \sin t C'_l(0) \}; \quad l = 1, 2, \dots, n-1.
\end{aligned} \tag{2-14}$$

Then, by using (2-9), one can obtain for  $l = 0, 1, \dots, n-1$

$$\begin{aligned}
C_l(t) &\approx \frac{e^{-i(1-\xi)t}}{n} \sum_{k=0}^{n-1} (e^{-i\xi nt} - 1 + n\delta_{k,l}) (\cos t C_k(0) - i \sin t A_k(0)), \\
A_l(t) &\approx \frac{e^{-i(1-\xi)t}}{n} \sum_{k=0}^{n-1} (e^{-i\xi nt} - 1 + n\delta_{k,l}) (\cos t A_k(0) - i \sin t C_k(0)),
\end{aligned} \tag{2-15}$$

It could be noted that for times such that  $\xi nt \ll 1$ , also for times such that  $\xi t = \frac{2k\pi}{n}$  with  $k \in \mathbb{Z}$ , the above result leads to  $C_l(t) \cong e^{-i(1-\xi)t} (\cos t C_l(0) - i \sin t A_l(0))$  and  $A_l(t) \cong e^{-i(1-\xi)t} (\cos t A_l(0) - i \sin t C_l(0))$ , so that we have  $|C_l(t)|^2 + |A_l(t)|^2 = |C_l(0)|^2 + |A_l(0)|^2$  and so, there is no exchange between the cavities. On the other hand, for the times such that

$\xi t = \frac{(2l+1)\pi}{n}$ , with  $l \in Z$ , the exchange between the cavities (excitation or photon transfer) can be achieved. For instance, in the case of two cavities  $n = 2$ , for the initial state  $|\psi(0)\rangle = |a_0\rangle = |e, 0\rangle|g, 0\rangle$  with initial conditions  $A_0(0) = 1$  and  $C_0(0) = A_1(0) = C_1(0) = 0$ , by using (2-15), we obtain at times  $t \simeq \frac{(2l+1)\pi}{2\xi}$ ,  $C_0(t) = A_0(t) = 0$ ,  $C_1(t) = -ie^{-i(1-\xi)t} \sin t$  and  $A_1(t) = e^{-i(1-\xi)t} \cos t$ , so that we have  $|\psi(t)\rangle = e^{-i(1-\xi)t}(\cos t|g, 0\rangle|e, 0\rangle - i \sin t|g, 0\rangle|g, 2\rangle)$ .

The results of this section can be used in order to discuss about qubit state transfer, photon transition and entanglement generation between the atoms. In order to clarify that, how one can discuss these arguments, we will consider the special cases of two and three identical cavities in the next sections in details.

### 3 Two coupled cavities: the case $n = 2$

For two cavities ( $n = 2$ ), by using the relations (2-7)-(2-9), one can calculate

$$C_0(t) = \frac{C'_0 + C'_1}{\sqrt{2}} = \frac{e^{-it}}{\sqrt{1+\xi^2}} \{ [\sqrt{1+\xi^2} \cos \xi t \cos t \sqrt{1+\xi^2} - \xi \sin \xi t \sin t \sqrt{1+\xi^2}] C_0(0) - i[\sqrt{1+\xi^2} \sin \xi t \cos t \sqrt{1+\xi^2} + \xi \cos \xi t \sin t \sqrt{1+\xi^2}] C_1(0) - i \sin t \sqrt{1+\xi^2} (\cos \xi t A_0(0) - i \sin \xi t A_1(0)) \},$$

$$C_1(t) = \frac{C'_0 - C'_1}{\sqrt{2}} = \frac{e^{-it}}{\sqrt{1+\xi^2}} \{ -i[\sqrt{1+\xi^2} \sin \xi t \cos t \sqrt{1+\xi^2} + \xi \cos \xi t \sin t \sqrt{1+\xi^2}] C_0(0) + [\sqrt{1+\xi^2} \cos \xi t \cos t \sqrt{1+\xi^2} - \xi \sin \xi t \sin t \sqrt{1+\xi^2}] C_1(0) - i \sin t \sqrt{1+\xi^2} (-i \sin \xi t A_0(0) + \cos \xi t A_1(0)) \},$$

$$A_0(t) = \frac{A'_0 + A'_1}{\sqrt{2}} = \frac{e^{-it}}{\sqrt{1+\xi^2}} \{ [\sqrt{1+\xi^2} \cos \xi t \cos t \sqrt{1+\xi^2} + \xi \sin \xi t \sin t \sqrt{1+\xi^2}] A_0(0) - i[\sqrt{1+\xi^2} \sin \xi t \cos t \sqrt{1+\xi^2} - \xi \cos \xi t \sin t \sqrt{1+\xi^2}] A_1(0) - i \sin t \sqrt{1+\xi^2} (\cos \xi t C_0(0) - i \sin \xi t C_1(0)) \},$$

$$A_1(t) = \frac{A'_0 - A'_1}{\sqrt{2}} = \frac{e^{-it}}{\sqrt{1+\xi^2}} \{ -i[\sqrt{1+\xi^2} \sin \xi t \cos t \sqrt{1+\xi^2} - \xi \cos \xi t \sin t \sqrt{1+\xi^2}] A_0(0) +$$

$$[\sqrt{1+\xi^2} \cos \xi t \cos t \sqrt{1+\xi^2} + \xi \sin \xi t \sin t \sqrt{1+\xi^2}] A_1(0) - i \sin t \sqrt{1+\xi^2} (-i \sin \xi t C_0(0) + \cos \xi t C_1(0))\}. \quad (3-16)$$

For instance, for the initial state  $|\psi\rangle = \frac{1}{\sqrt{2}}(|c_0\rangle + |c_1\rangle) = \frac{1}{\sqrt{2}}(|g, 2\rangle|g, 0\rangle + |g, 0\rangle|g, 2\rangle)$ , we have the initial conditions  $C_0(0) = C_1(0) = \frac{1}{\sqrt{2}}$  and  $A_0(0) = A_1(0) = 0$ . Then, by using the relations (3-16), the evolved state of the system will take the form

$$|\psi(t)\rangle = \frac{e^{-i(1+\xi)t}}{\sqrt{2(1+\xi^2)}} \{ [\sqrt{1+\xi^2} \cos \xi t \cos t \sqrt{1+\xi^2} - i \xi \sin t \sqrt{1+\xi^2}] (|c_0\rangle + |c_1\rangle) - i \sin t \sqrt{1+\xi^2} (|a_0\rangle + |a_1\rangle) \}.$$

Now, in order to investigate generation of entanglement between two atoms, we can evaluate the density matrix  $\rho_a$  associated with the atoms as

$$\rho_a(t) = Tr_c(|\psi(t)\rangle\langle\psi(t)|) = \frac{1}{2(1+\xi^2)} \{ 2(\xi^2 + \cos^2 t \sqrt{1+\xi^2}) (|g\rangle^{(1)}\langle g| \otimes |g\rangle^{(2)}\langle g|) + \sin^2 t \sqrt{1+\xi^2} (|e\rangle^{(1)}\langle e| \otimes |g\rangle^{(2)}\langle g| + |g\rangle^{(1)}\langle g| \otimes |e\rangle^{(2)}\langle e| + |e\rangle^{(1)}\langle g| \otimes |g\rangle^{(2)}\langle e| + |g\rangle^{(1)}\langle e| \otimes |e\rangle^{(2)}\langle g|) \}$$

where,  $Tr_c$  denotes the partial trace over the photonic states  $|2, 0\rangle$ ,  $|0, 2\rangle$  and  $|0, 0\rangle$ . Now, for a given hopping parameter  $\xi$ , one can use the Peres-Horodecki criteria [24, 25] known also as positive partial transpose (PPT) criteria, in order to determine that for which times  $t$ , the state  $\rho_a(t)$  is entangled, and particularly we can obtain the time  $T$  at which the perfect transfer of photonic entanglement to the atomic one, is achieved. To this end, we choose the order of atomic basis as  $|g, g\rangle$ ,  $|g, e\rangle$ ,  $|e, g\rangle$  and  $|e, e\rangle$ , so that the partial transpose of the atomic state takes the following matrix form

$$(\rho_a(t))^{T_1} = \begin{pmatrix} 2(\xi^2 + \cos^2 t \sqrt{1+\xi^2}) & 0 & 0 & \sin^2 t \sqrt{1+\xi^2} \\ 0 & \sin^2 t \sqrt{1+\xi^2} & 0 & 0 \\ 0 & 0 & \sin^2 t \sqrt{1+\xi^2} & 0 \\ \sin^2 t \sqrt{1+\xi^2} & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding eigenvalues of  $(\rho_a(t))^{T_1}$  are given by  $\lambda = \sin^2 t \sqrt{1+\xi^2}$  with double degeneracy, and  $\lambda_{\pm} = (\xi^2 + \cos^2 t \sqrt{1+\xi^2}) \pm \sqrt{(\xi^2 + \cos^2 t \sqrt{1+\xi^2})^2 + \sin^4 t \sqrt{1+\xi^2}}$ . Therefore, the eigenvalue  $\lambda_-$  is clearly negative, except for times  $t = \frac{k\pi}{\sqrt{1+\xi^2}}$ ,  $k \in Z$ , where the atomic state

$\rho_a$  is separable. In order to evaluate the amount of entanglement of the atomic state  $\rho_a$ , one can calculate the corresponding concurrence [26], as  $C(\rho_a(t)) = \frac{\sin^2 t \sqrt{1+\xi^2}}{1+\xi^2}$ . Then, for times  $T = \frac{(2l+1)\pi}{2\sqrt{1+\xi^2}}$ ,  $l \in Z$ , (or in the suitable units,  $T = \frac{(2l+1)\pi}{2\sqrt{1+\xi'^2}}$  with  $\xi' = \frac{2\hbar\xi}{E_0^+ - E_0^-}$ ) the maximum value of the atomic entanglement is achieved and the corresponding concurrence takes its maximum value  $C_{max} = \frac{1}{1+\xi^2}$ , where the atomic density matrix will be maximally entangled for small hopping  $\xi \rightarrow 0$  (see Fig.1 for different values of  $\xi$ ). Moreover, this result indicates that, for large hopping strength  $\xi \rightarrow \infty$ , we have  $C(\rho_a(t)) \rightarrow 0$  and so, the initially unentangled atoms, remain effectively unentangled for all the next times.

Now, consider the initial state  $|\psi(0)\rangle = |e, 0\rangle|g, 0\rangle$  with  $C_0(0) = C_1(0) = A_1(0) = 0$  and  $A_0(0) = 1$ . The equations (3-16) give

$$\begin{aligned} C_0(t) &= \frac{-ie^{-it}}{\sqrt{1+\xi^2}} \sin t \sqrt{1+\xi^2} \cos \xi t, \\ C_1(t) &= \frac{-e^{-it}}{\sqrt{1+\xi^2}} \sin t \sqrt{1+\xi^2} \sin \xi t, \\ A_0(t) &= \frac{e^{-it}}{\sqrt{1+\xi^2}} \{ \sqrt{1+\xi^2} \cos t \sqrt{1+\xi^2} \cos \xi t + \xi \sin t \sqrt{1+\xi^2} \sin \xi t \}, \\ A_1(t) &= \frac{-ie^{-it}}{\sqrt{1+\xi^2}} \{ \sqrt{1+\xi^2} \cos t \sqrt{1+\xi^2} \sin \xi t - \xi \sin t \sqrt{1+\xi^2} \cos \xi t \}, \end{aligned}$$

so that, one obtains

$$\begin{aligned} |C_0(t)|^2 + |C_1(t)|^2 &= \frac{\sin^2 t \sqrt{1+\xi^2}}{1+\xi^2}, \\ |A_0(t)|^2 + |A_1(t)|^2 &= \frac{\xi^2 + \cos^2 t \sqrt{1+\xi^2}}{1+\xi^2}. \end{aligned}$$

Therefore, in the limit of large hopping  $\xi$ , we have  $|A_0(t)|^2 + |A_1(t)|^2 \rightarrow 1$  and so the only effective terms which survive, are  $A_0(t)$  and  $A_1(t)$ .

In addition, one can discuss two photon transfer from the first cavity to the second one. To do so, we consider the initial state  $|\psi(0)\rangle = |g, 2\rangle|g, 0\rangle$  with initial conditions  $C_0(0) = 1$ ,  $C_1(0) = A_0(0) = A_1(0) = 0$ . Then, the equations (3-16) give

$$C_0(t) = \frac{e^{-it}}{\sqrt{1+\xi^2}} \{ \sqrt{1+\xi^2} \cos \xi t \cos t \sqrt{1+\xi^2} - \xi \sin \xi t \sin t \sqrt{1+\xi^2} \},$$

$$\begin{aligned}
C_1(t) &= \frac{-ie^{-it}}{\sqrt{1+\xi^2}} \{ \sqrt{1+\xi^2} \sin \xi t \cos t \sqrt{1+\xi^2} + \xi \cos \xi t \sin t \sqrt{1+\xi^2} \}, \\
A_0(t) &= \frac{-ie^{-it}}{\sqrt{1+\xi^2}} \cos \xi t \sin t \sqrt{1+\xi^2}, \\
A_1(t) &= \frac{-e^{-it}}{\sqrt{1+\xi^2}} \sin \xi t \sin t \sqrt{1+\xi^2}.
\end{aligned}$$

Now, for large enough  $\xi \gg$ , we obtain

$$C_0(t) = e^{-it} \cos 2\xi t, \quad C_1(t) = -ie^{-it} \sin 2\xi t, \quad A_0(t) = \frac{-ie^{-it} \sin 2\xi t}{2\xi} \cong 0, \quad A_1(t) = \frac{-e^{-it} \sin^2 \xi t}{\xi} \cong 0.$$

Then, after times  $T = \frac{(2k+1)\pi}{4\xi}$  with  $k \in \mathbb{Z}$ , we have  $|\psi(T)\rangle = (-1)^{k+1} i e^{-it} |g, 0\rangle |g, 2\rangle$  with  $|C_1(T)|^2 = 1$  and so, two photons of the first cavity are transmitted to the other cavity, perfectly.

## 4 Three coupled cavities: Large and Small hoppings

The case  $n = 3$  identical cavities can be considered similar to the case of  $n = 2$ , by using the relations (2-7)-(2-9). Here we consider only the limits of large and small hopping  $\xi$ .

### 4.1 Large hopping $\xi \gg$

In the limit of large  $\xi$ , we use the equation (2-12) to obtain

$$\begin{aligned}
C_0(t) &\approx \frac{1}{3} \{ [C_0(0) + C_1(0) + C_2(0)] e^{-4i\xi t} + [2C_0(0) - C_1(0) - C_2(0)] e^{2i\xi t} \}, \\
C_1(t) &\approx \frac{1}{3} \{ [C_0(0) + C_1(0) + C_2(0)] e^{-4i\xi t} + [-C_0(0) + 2C_1(0) - C_2(0)] e^{2i\xi t} \}, \\
C_2(t) &\approx \frac{1}{3} \{ [C_0(0) + C_1(0) + C_2(0)] e^{-4i\xi t} + [-C_0(0) - C_1(0) + 2C_2(0)] e^{2i\xi t} \}, \\
A_l(t) &\approx A_l(0), \quad \text{for} \quad l = 0, 1, 2.
\end{aligned}$$

The above results are in correspondence with those of Ref.[18]. By considering the initial state  $|\psi(0)\rangle = |c_0\rangle = |g, 2\rangle |g, 0\rangle |g, 0\rangle$  with initial conditions  $C_0(0) = 1, C_1(0) = C_2(0) = A_0(0) =$

$A_1(0) = A_2(0) = 0$ , we obtain the evolved state after time  $t$  as

$$|\psi(t)\rangle \approx \frac{1}{3}\{(e^{-4i\xi t} + 2e^{2i\xi t})|c_0\rangle + (e^{-4i\xi t} - e^{2i\xi t})(|c_1\rangle + |c_2\rangle)\}.$$

Then, the probability of observing two photons at the first cavity ( $|C_0(t)|^2$ ) and that of observing two photons at the two other cavities ( $|C_1(t)|^2 = |C_2(t)|^2$ ), are given respectively by

$$|C_0(t)|^2 = \frac{1}{9}[1 + 4(1 + \cos 6\xi t)], \quad |C_1(t)|^2 = |C_2(t)|^2 = \frac{2}{9}[1 - \cos 6\xi t],$$

which indicates that for times  $T = \frac{(2k+1)\pi}{6\xi}$ , with  $k \in Z$ , the corresponding two photons initially located at the first cavity, are transmitted to one of the other cavities with equal probability  $\frac{4}{9}$ .

## 4.2 Small hopping $\xi \ll$

In the limit of small hopping  $\xi \ll$ , the equation (2-15) leads to the following results for three identical cavities:

$$\begin{aligned} C_0(t) &\approx \frac{e^{-i(1-\xi)t}}{3}\{(e^{-3i\xi t} + 2)[\cos t C_0(0) - i \sin t A_0(0)] + (e^{-3i\xi t} - 1)[\cos t(C_1(0) + C_2(0)) - i \sin t(A_1(0) + A_2(0))]\}, \\ C_1(t) &\approx \frac{e^{-i(1-\xi)t}}{3}\{(e^{-3i\xi t} + 2)[\cos t C_1(0) - i \sin t A_1(0)] + (e^{-3i\xi t} - 1)[\cos t(C_0(0) + C_2(0)) - i \sin t(A_0(0) + A_2(0))]\}, \\ C_2(t) &\approx \frac{e^{-i(1-\xi)t}}{3}\{(e^{-3i\xi t} + 2)[\cos t C_2(0) - i \sin t A_2(0)] + (e^{-3i\xi t} - 1)[\cos t(C_0(0) + C_1(0)) - i \sin t(A_0(0) + A_1(0))]\}, \\ A_0(t) &\approx \frac{e^{-i(1-\xi)t}}{3}\{(e^{-3i\xi t} + 2)[\cos t A_0(0) - i \sin t C_0(0)] + (e^{-3i\xi t} - 1)[\cos t(A_1(0) + A_2(0)) - i \sin t(C_1(0) + C_2(0))]\}, \\ A_1(t) &\approx \frac{e^{-i(1-\xi)t}}{3}\{(e^{-3i\xi t} + 2)[\cos t A_1(0) - i \sin t C_1(0)] + (e^{-3i\xi t} - 1)[\cos t(A_0(0) + A_2(0)) - i \sin t(C_0(0) + C_2(0))]\}, \\ A_2(t) &\approx \frac{e^{-i(1-\xi)t}}{3}\{(e^{-3i\xi t} + 2)[\cos t A_2(0) - i \sin t C_2(0)] + (e^{-3i\xi t} - 1)[\cos t(A_0(0) + A_1(0)) - i \sin t(C_0(0) + C_1(0))]\}. \end{aligned}$$

Now, by considering for example the initial state  $|\psi(0)\rangle = |a_0\rangle = |e, 0\rangle|g, 0\rangle|g, 0\rangle$ , with initial conditions  $A_0(0) = 1$  and  $C_0(0) = C_1(0) = C_2(0) = A_1(0) = A_2(0) = 0$ , we obtain

$$C_0(t) \approx \frac{-i \sin t e^{-i(1-\xi)t} e^{-i(1-\xi)t}}{3}(e^{-3i\xi t} + 2), \quad C_1(t) = C_2(t) \approx \frac{-i \sin t e^{-i(1-\xi)t} e^{-i(1-\xi)t}}{3}(e^{-3i\xi t} - 1),$$

$$A_0(t) \approx \frac{\cos t e^{-i(1-\xi)t}}{3}(e^{-3i\xi t} + 2), \quad A_1(t) = A_2(t) \approx \frac{\cos t e^{-i(1-\xi)t}}{3}(e^{-3i\xi t} - 1).$$

Therefore, after times  $t = k\pi$ , with  $k \in \mathbb{Z}$ , the probability amplitudes  $C_0(t)$ ,  $C_1(t)$  and  $C_2(t)$  will be zero and we will have  $A_0(k\pi) \approx \frac{(-1)^k e^{-i(1-\xi)k\pi}}{3}(e^{-3i\xi k\pi} + 2)$  and  $A_1(k\pi) = A_2(k\pi) \approx \frac{(-1)^k e^{-i(1-\xi)k\pi}}{3}(e^{-3i\xi k\pi} - 1)$ . This indicates that, by choosing the hopping strength as  $\xi = \frac{2l+1}{3k}$  with large  $k \in \mathbb{Z}$  and small  $l \in \mathbb{Z}$ , the excitation of the first atom located at the first cavity, can be transmitted with equal probability  $\frac{4}{9}$ , to one of the other two atoms.

## 5 Conclusion

In summery, the quantum entanglement properties of  $n$  coupled atom-cavity systems via two-photon exchange interaction, was analyzed. By employing the photon number operator symmetry of the Hamiltonian, the corresponding Hilbert space's dimension was reduced from  $3^n$  to  $2n$  and then by introducing some suitable Fourier transformed basis states for the state space of the system, the corresponding Hamiltonian was block-diagonalized with 2 dimensional blocks. Due to this useful reduction, the corresponding Shrödinger equation was solved exactly for any number  $n$  of atom-cavities, and excitation and photon transition between the atoms and the cavities was discussed. The perfect transfer time and the times at which the maximal quantum entanglement can be periodically generated between the initially unentangled atoms are obtained in terms of the hopping parameter (coupling strength) between the cavities. The large and small hopping limits was discussed where, it was shown that for large hopping strength, the initially unentangled atoms remain effectively unentangled forever.

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**Figure Caption**

**Figure.1:** Shows the concurrence  $C(\rho)$  of the atomic state  $\rho_a(t)$  with respect to time, for different values of hopping strength (a)  $\xi = 0.1$ , (b)  $\xi = 0.5$ , (c)  $\xi = 0.9$  and (d)  $\xi = 2$ .